CONTINUOUS PROBABILITY DISTRIBUTION

This is a distribution which takes on any value within a given interval.

Summary:

A probability density function **(p·d·f)** is a function that defines the probability of an event to occur.

A continuous p·d·f ***f(x)*** defined over the interval $\mathbf{a \le x \le b}$ is such that**:**

(a) The total area under ***f(x)*** = sum of all probabilities = **1.**

$$\Rightarrow \int_a^b f(x)\,dx = 1.$$

(b) $P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx.$

NOTE:

(i) The values of $P(x_1 < X < x_2), P(x_1 \le X < x_2), P(x_1 < X \le x_2)$ and $P(x_1 \le X \le x_2)$ are the same.

(ii) The values $P(X = x_1) = P(X = x_2) = 0$ since **X** deals with a range of values

(c) Expectation**,** $E(X) = \int_a^b x f(x)\,dx\,.$

(d) Variance, $\mathbf{Var(X) = E(X^2) - E^2(X)}$.

where $$\mathbf{E(X^2) = \int_a^b x^2 f(x)\, dx}, \qquad \mathbf{E^2(X) = [E(X)]^2}.$$

(e) Standard deviation $\boldsymbol{\sigma} = \sqrt{\textbf{variance}}$

(f) For a continuous r·v **X** and constants **a** and **b,**

(i) E(a) = a	**Var(a) = 0**
(ii) E(aX) = aE(X)	$\mathbf{Var(aX) = a^2Var(X)}$
(iii) E(aX + b) = aE(X) + b	$\mathbf{Var(aX + b) = a^2Var(X)}$

(g) Median is the value **m** which satisfies the relation $$\int_a^m f(x)\, \mathbf{dx} = \mathbf{0 \cdot 5}.$$

I·e The median encloses an area of **0·5** below it.

(h) Lower quartile is the value $\mathbf{q_1}$ which satisfies the relation $$\int_a^{q_1} f(x)\, \mathbf{dx} = \mathbf{0 \cdot 25}.$$

I·e The lower quartile encloses an area of **0·25** below it.

(I) Upper quartile is the value $\mathbf{q_3}$ which satisfies the relation $$\int_a^{q_3} f(x)\, \mathbf{dx} = \mathbf{0 \cdot 75}.$$

I·e The upper quartile encloses an area of **0·75** below it.

(j) Interquartile range $= \mathbf{Q_3 - Q_1}$

(k) The J^{th} percentile is the value **p** that satisfies the relation $\int_a^p f(x)\,dx = \frac{J}{100}$.

NOTE: The median, quartiles and percentiles of a p·d·f defined over different intervals are obtained by first investigating the interval in which they are located.

(L) The graph of the p·d·f ***f(x)*** can either be linear or a curve.

(m) Mode is value of **x** at which the p·d·f ***f(x)*** attains its maximum value.

The graph of ***f(x)*** gives the location of the mode. If the p·d·f ***f(x)*** is non linear, the value **x** at which ***f(x)*** has a maximum value occurs when $f^{1}(x) = 0$ provided $f^{11}(x) < 0$.

EXAMPLES:

1. The p·d·f of a continuous **r·v X** is given by:

$$f(x) = \begin{cases} \beta x & , \quad 0 \le x \le 1 \\ \frac{1}{2}\beta(3 - x) & , \quad 1 < x < 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find:

(i) the value of β

(ii) $P(X = 2)$

(iii) $P(X < 2)$

(iv) $P(X > 1 \cdot 4)$

(v) $\mathbf{P(0\cdot 8 < X < 2)}$

(vi) $\mathbf{P(|X - 1| < 0\cdot 6)}$

(vii) $\mathbf{P(|X - 1| > 0\cdot 6)}$

(viii) $\mathbf{P(0\cdot 2 < X < 2\cdot 5 / X \geq 0\cdot 7)}$

(ix) the mode, mean and standard deviation of **X.**

(x) E(3X + 5)

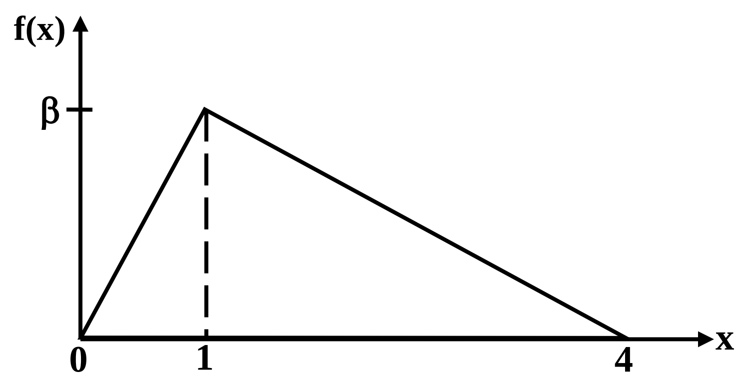
(xi) Var(3X + 5)

(xii) the median and semi-interquartile range of **X.**

(xiii) the value of **b** such that $\mathbf{P(X \leq b) = 0\cdot 6.}$

(xiv) the **30^{th}** to **80^{th}** percentile range of **X.**

2. The p·d·f ***f(x)*** of a **r·v X** takes on the form shown in the sketch below**:**



Find the**:**

(i) value of **β**

(ii) equations of the p·d·f

(iii) $\mathbf{P(0\cdot 5 < X < 2)}$

(iv) mean of **X.**

(v) median of **X.**

3. The p·d·f of a continuous **r·v X** is such that**:**

$$f(x) = \begin{cases} \beta x(6-x)^2 , & 0 < x < 6 \\ 0 , & \text{otherwise} \end{cases}$$

Find the**:**

(i) value of **β**

(ii) mode of **X**

(iii) mean of **X**

4. The outputs of **9** machines in a factory are independent random variables each with probability density function given by

$$f(x) = \begin{cases} \beta x , & 0 \le x \le 10 \\ \beta(20-x) , & 10 \le x \le 20 \\ 0 , & \text{otherwise} \end{cases}$$

Find the**:**

(i) value of **β.**

(ii) expected value and variance of the output of each machine. Hence or otherwise find the expected value and variance of the total output from all machines.

5. The mass **X kg** of maize flour produced per hour is modeled by a continuous r·v whose p·d·f is given by**:**

$$f(x) = \begin{cases} \beta x^2 & , \quad 0 \le x \le 2 \\ \beta(6 - x) & , \quad 2 \le x \le 6 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch the p·d·f of **X.** Hence state the mode of **X**

(b) Find the**:**

(i) value of β

(ii) $P(|X - 3| < 2)$

(iii) mean mass produced per hour

(c) Given that maize flour is sold at **sh 2400** per kg and the cost of running the production is **sh 200** per hour, taking **shs Y** as the profit made per hour.

(i) Express **Y** in terms of **X.**

(ii) Find the expected value of **Y.**

6. A r·v **X** has the following p·d·f

$$f(x) = \begin{cases} \beta x & , \quad 1 \le x \le 3 \\ \lambda(4 - x) & , \quad 3 < x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Show that $\lambda = 3\beta$.

(b) Find **:**

(i) the values of $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$

(ii) the mean and variance of **X**

(iii) the median of **X**

(iv) $\mathbf{P(3 \leq X \leq 4/X > 2)}$

7. A r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} \frac{2}{13}(x+1) & , \ 0 \leq x \leq a \\ \frac{2}{13}(5-x) & , \ a < x < b \\ 0 & , \ \text{otherwise} \end{cases}$$

Find the**:**

(i) values of **a** and **b**.

(ii) median of **X.**

(iii) $\mathbf{P(X > 0{\cdot}5/0{\cdot}25 \leq X \leq 1)}$

CUMULATIVE DISTRIBUTION FUNCTION *F(x)*

This function gives the accumulated probability up to **x.** It is obtained by integrating the p·d·f as follows: $\mathbf{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt.}$

The cumulative distribution function is sometimes known as a distribution function

PROPERTIES OF *F(x)*

(i) F(x) must be defined over the interval $-\infty \leq \mathbf{x} \leq \infty$.

(ii) $\mathbf{0 \leq F(x) \leq 1,}$ for all values of **x.**

(iii) $\mathbf{P(x_1 < X < x_2) = P(X \leq x_2) - P(X \leq x_1) = F(x_2) - F(x_1)}$

(iv) The median, **m,** lower quartile**,** $\mathbf{q_1}$**,** and upper quartile $\mathbf{q_3}$ are the values for which $\mathbf{F(m) = \frac{1}{2}}$**,** $\mathbf{F(q_1) = \frac{1}{4}}$ and $\mathbf{F(q_2) = \frac{3}{4}}$ respectively.

(v) $\mathbf{P(X \leq x) + P(X > x) = 1}$

$\therefore$ The complementary cumulative distribution function

$$\mathbf{P(X > x) = 1 - P(X \leq x) = 1 - F(x)}$$

(vi) The p·d·f ***f(x)*** can be obtained by differentiating the cumulative distribution

$$\therefore \quad F^{1}(x) = \text{p·d·f } f(x)$$

EXAMPLES:

1. The p·d·f of a continuous **r·v X** is given by**:**

$$f(x) = \begin{cases} \beta(x-1) & , \ 1 < x \le 3 \\ \frac{1}{2}\beta(7-x) & , \ 3 < x \le 7 \\ 0 & , \ \text{otherwise} \end{cases}$$

Find**:**

(i) the value of **β,** hence ***f(x)***

(ii) the cumulative distribution function **F(x)** and sketch it.

(iii) $P(2\cdot8 < X < 5\cdot2)$

(iv) $P(X > 4)$

(v) the median of **X.**

(vi) the interquartile range of X

(vii) the **20th** percentile of **X.**

Solution:

(ii) Note: F(x) is concave up parabola over the interval $1 < x \le 3$ and concave down parabola over the interval $3 < x \le 7$.

2. A continuous r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} k(3-x) & , \quad 1 \le x \le 2 \\ k & , \quad 2 < x \le 3 \\ k(x-2) & , \quad 3 < x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x),*** hence deduce the mean and median of **X.**

(b) Find**:**

(i) the value of ***k*.**

(ii) the cumulative distribution function **F(x)** and sketch it.

(iii) $P(X \ge 3\cdot5 / 3 \le X \le 4)$

3. The distribution function of a continuous **r·v** ***X*** is as follows**:**

$$P(X \le x) = \begin{cases} 0 & , \quad x \le 1 \\ \frac{1}{12}(x-1)^2 & , \quad 1 < x \le 3 \\ \frac{1}{24}(\beta x + \lambda - x^2) & , \quad 3 < x \le 7 \\ 1 & , \quad x > 7 \end{cases}$$

Find**:**

(i) the values of β and λ

(ii) $P(X \le 4)$

(iii) the median of ***X***

(iii) the *p·d·f* of ***X***

(iv) the mean, μ of the distribution

(v) $P(|X - \mu| > 0\cdot8)$

4. The cumulative distribution of a continuous **r·v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ 2x - 2x^2 & , \quad 0 \le x \le 0{\cdot}25 \\ \beta + x & , \quad 0{\cdot}25 < x \le 0{\cdot}5 \\ \alpha + 2x^2 - x & , \quad 0{\cdot}5 < x \le 0{\cdot}75 \\ 1 & , \quad x > 0{\cdot}75 \end{cases}$$

Find**:**

(i) the values of β and α**,** hence sketch **F(x).**

(ii) $P(|X - 0{\cdot}375| < 0{\cdot}25)$

(iii) the p·d·f of **X** and sketch it**,** hence deduce the mean and median of **X.**

5. A continuous r·v **X** is distributed as follows**:**

$$P(X > x) = a + bx^3 \quad , \quad 0 \le x \le 4$$

(i) By first finding the cumulative distribution of **X** or otherwise, find the values of **a** and **b.**

(ii) Show that **E(X) = 3,** and find the standard deviation **σ** of **X.**

UNIFORM DISTRIBUTION

This distribution is sometimes called a rectangular distribution.

Summary:

If a **r·v X** is uniformly distributed over the interval **[a, b],** then**:**

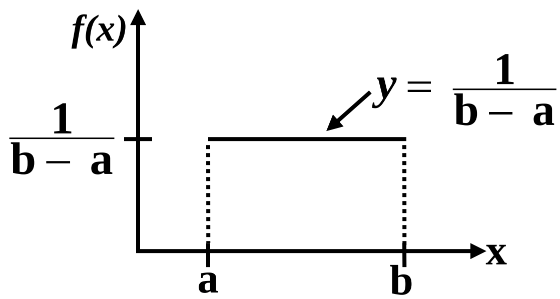
(i) the p·d·f of **X** is given by**:** $f(x) = \begin{cases} \dfrac{1}{b-a} & , \ a \le x \le b \\ 0 & , \ \text{otherwise} \end{cases}$

(ii) the mean of **X** is $\dfrac{a+b}{2}$**.**

(iii) the variance of **X** is $\dfrac{(b-a)^2}{12}$**.**

(iv) the median of **X** is $\dfrac{a+b}{2}$**.**

(v) the graph of *f(x)* is as follows**:**



EXAMPLES:

1. A **r·v X** is uniformly distributed over the interval **[a, b].**

(a) State the p·d·f of **X** and sketch it**.**

(b) Show that**:**

(i) the mean of **X** is $\dfrac{a+b}{2}$**.**

(ii) the variance of **X** is $\dfrac{(b-a)^2}{12}$**.**

(iii) the median of **X** is $\dfrac{a+b}{2}$**.**

(c) Find the cumulative distribution function of **X** and sketch it.

2. (a) A **r·v X** is uniformly distributed over the interval **[2, 5].** Find

$$\mathbf{P(X \geq 2\cdot 3 / X \leq 4\cdot 5)}$$

(b) The number of vehicles crossing a roundabout take on a **r·v X** with uniformly distribution over the interval $[\,x_1\,, x_2\,]$. If the expected number of vehicles crossing the roundabout is **15** with variance **3,** calculate the**:**

(i) values of x_1 and x_2.

(ii) probability that at least **14** vehicles cross the roundabout**.**

(iii) probability that the number of vehicles crossing the roundabout lies within one standard deviation of the mean**.**

EER:

1. The p·d·f of a continuous r·v **X** is given by

$$f(x) = \begin{cases} \beta(1-x^2) & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find**:**

(i) the value of β.

(ii) the mean, **μ** and standard deviation, **σ** of **X.**

(iii) E(8X + 3) and **Var(8X + 3)**

(iv) $\mathbf{P(|X - \mu| \leq \sigma)}$

[Ans: (i) 1·5 (ii) 0·375, 0·2437 (iii) 6, 3·8 (iii) 0·6145]

2. The p·d·f of a continuous r·v **X** is given by

$$f(x) = \begin{cases} \mathbf{\frac{2}{3}x} & , \quad \mathbf{0 \le x \le 1} \\ \mathbf{\frac{1}{3}(3-x)} & , \quad \mathbf{1 \le x < 3} \\ \mathbf{0} & , \quad \textbf{otherwise} \end{cases}$$

(a) Show that ***f(x)*** represents a probability density function**.**

(b) Find the**:**

(i) median of **X**

(ii) 80th percentile of **X.**

(iii) value of **b** such that $\mathbf{P(X \le b) = 0 \cdot 6}$**.**

(iv) expressions for $\mathbf{P(X \le x)}$ and sketch it.

[Ans: b(i) 1·2679 (ii) 1·9046 (iii) 1·4508]

3. The p·d·f of a continuous r·v **X** is given by

$$f(x) = \begin{cases} \beta \mathbf{x} - \alpha \mathbf{x^2} & , \quad \mathbf{0 \le x \le 2} \\ \mathbf{0} & , \quad \textbf{otherwise} \end{cases}$$

Given that the mean of **X** is **1,** find the**:**

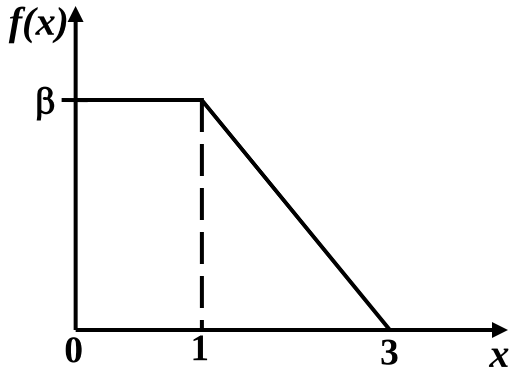
(i) values of β and α**.**

(ii) variance of **X**

(iii) mode of **X**

[Ans: (i) 1·5 , 0·75 (ii) 0·2 (iii) 1]

4. The p·d·f ***f(x)*** of a **r·v X** takes on the form shown in the sketch below**:**



Find**:**

(i) the value of **β**

(ii) the equations of the p·d·f

(iii) the mean of **X.**

(iv) $\mathbf{P(|X - 1\cdot25| < 0\cdot75)}$

(v) $\mathbf{P(0\cdot2 < X < 1\cdot5 / X \geq 0\cdot6)}$

[Ans: (i) $\frac{1}{2}$ (iii) 1·0833 (iv) 0·625 (v) 0·5982]

5. A r·v **X** is uniformly distributed with mean **7·5** and variance **0·75** over the interval **[a, b].** Find**:**

(i) the values of **a** and **b.**

(ii) the p·d·f of the distribution**.**

(iii) $\mathbf{P(7\cdot2 \leq X \leq 8\cdot4)}$

(iv) 80th percentile of **X.**

(v) probability that **X** lies within one standard deviation of the mean**.**

(vi) cumulative distribution function of **X.**

[Ans: (i) 6, 9 (iii) 0·4 (iv) 8·4 (v) 0·5774]

6. The time taken to perform a particular task **t hours** is given by the p·d·f**:**

$$f(t) = \begin{cases} \mathbf{10\beta t^2} & , \quad \mathbf{0 \le t \le 0 \cdot 6} \\ \mathbf{9\beta(1-t)} & , \quad \mathbf{0 \cdot 6 < t < 1} \\ \mathbf{0} & , \quad \textbf{otherwise} \end{cases}$$

(a) Find the**:**

(i) value of **β**

(ii) most likely time**.**

(iii) expected time**.**

(b) Determine the probability that the time will be**:**

(i) more than **48** minutes**.**

(ii) between **24**and **48** minutes**.**

[Ans: (a) (i) $\frac{25}{36}$ (ii) 0·6 (iii) 0·591 (b) (i) 0·125 (ii) 0·727]

7. A continuous r·v **X** has the following p·d·f

$$f(x) = \begin{cases} \beta\left(x - \frac{1}{k}\right) & , \quad 0 \le x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Given that **P(X > 1) = 0·8,** find the**:**

(i) values of k and β.

(ii) probability that **X** lies between **0·5** and **2·5**

(iii) mean of **X**

[Ans: (i) –1, $^2/_{15}$ (ii) 0·6667 (iii) 1·8]

8. A continuous **r·v X** has the following p·d·f **:**

$$f(x) = \begin{cases} \beta x^2 & , \quad 0 \le x \le 2 \\ \beta(6-x) & , \quad 2 \le x \le 6 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x)***

(b) Find**:**

(i) the value of β.

(ii) the median of **X.**

(iii) $P(|X - 2{\cdot}75| < 1{\cdot}25)$

[Ans: b(i) $\frac{3}{32}$ (ii) 2·734 (iii) 0·7070]

9. A continuous r·v **X** has the following p·d·f

$$f(x) = \begin{cases} k & , \quad 0 \le x \le 2 \\ k(2x - 3) & , \quad 2 \le x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x)***

(b) Find**:**

(i) the value of **k.**

(ii) the mean and variance of **X.**

(iii) $P(X \ge 2{\cdot}5)$

(iv) $P(1 \le X \le 2{\cdot}5)$

(v) $P(0 < X < 2 / X \ge 1)$

[Ans: (i) $\frac{1}{4}$ (ii) $\frac{43}{24}$, 0·8316 (iii) 0·3125 (iv) 0·4375 (v) $\frac{1}{3}$]

10. The cumulative distribution of a continuous **r·v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x \le 1 \\ \beta(x-1)^2 & , \quad 1 < x \le 3 \\ \alpha(14x - x^2 - 25) & , \quad 3 < x \le 7 \\ 1 & , \quad x > 7 \end{cases}$$

Find**:**

(i) the values of β and α**.**

(ii) $P(2\cdot 8 \le X \le 5\cdot 2)$

(iii) the median of **X**

(iv) the p·d·f of **X** and sketch it**.**

(v) the mean, **μ** of **X.**

(vi) $P(|X - \mu| > 0\cdot 8)$

[Ans: (i) $\frac{1}{12}$, $\frac{1}{24}$ (ii) 0·595 (iii) 3·45 (v) $\frac{11}{3}$ (vi) 0·5578]

11. The number of boats **X** crossing a river is uniformly distributed between **150** and **210.**

(a) State the p·d·f of the distribution**.**

(b) Find the**:**

(i) probability that between **170** and **194** boats cross the river**.**

(ii) expected number of boats to cross the river**.**

(iii) standard deviation for the distribution**.**

[Ans: b(i) 0·4 (ii) 180 (iii) 7·7460]

12. The cumulative distribution of a continuous **r·v X** is such that**:**

$$\mathbf{F(x)} = \begin{cases} \mathbf{0} & , \quad \mathbf{x \le 1} \\ \mathbf{0 \cdot 5(x^2 - 2x + 1)} & , \quad \mathbf{1 < x \le 2} \\ \mathbf{\beta x + \lambda x^2 - 3 \cdot 5} & , \quad \mathbf{2 < x \le 3} \\ \mathbf{1} & , \quad \mathbf{x > 3} \end{cases}$$

Find**:**

(i) the values of β and λ**.** Hence sketch **F(x)**

(ii) $\mathbf{P(1 \cdot 5 < X < 2 \cdot 5 / X > 2)}$

(iii) the p·d·f of **X** and sketch it**.** Hence deduce the mean, mode and median

[Ans: (i) 3 , –0·5 (ii) 0·75 (iii) 2, 2, 2]

13. A continuous r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} k(3 - x) & , \quad 1 \le x \le 2 \\ k & , \quad 2 < x \le 3 \\ k(x - 2) & , \quad 3 < x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x)*,** hence deduce the mean and median of **X.**

(b) Find**:**

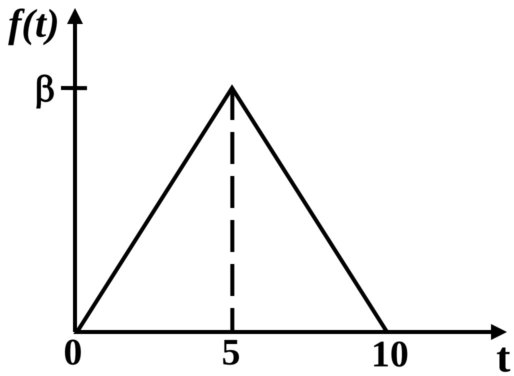
(i) the value of ***k*.**

(ii) $\mathbf{P(X \ge 3 \cdot 5 / 3 \le X \le 4)}$

(iii) the $\mathbf{80^{th}}$ percentile of **X.**

[Ans: (a) 2·5, 2·5 b(i) 0·25 (ii) 0·583 (iii) 3·5492]

14. The time**, T,** taken to complete a certain task can be modeled as in the diagram below, where **t** is the time in minutes.



Determine the:

(i) value of **β**

(ii) equations of the p·d·f

(iii) E(T)

(iv) probability that the task will be completed between **4** and **7** minutes.

(v) probability that the task will be completed in less than **2** minutes

[Ans: (i) 0·2 (iii) 5 (iv) 0·5 (v) 0·08]

15. A r·v **X** is uniformly distributed with variance **6·75** over the interval $3 \leq x \leq b$**.** Find**:**

(i) the value of **b.**

(ii) the p·d·f of the distribution**.**

(iii) $P(5 < X < 9 / X \geq 7)$

[Ans: (i) 12 (iii) 0·4]

16. A **r·v X** is uniformly distributed over the interval **[a, b].**

(a) State the p·d·f of **X** and sketch it**.**

(b) Show that**:**

(i) the mean of **X** is $\frac{a+b}{2}$.

(ii) the variance of **X** is $\frac{(b-a)^2}{12}$.

(iii) the median of **X** is $\frac{a+b}{2}$.

(c) Find the cumulative distribution function of **X** and sketch it.

17. A r·v **X** has the following cumulative distribution function

$$F(x) = \begin{cases} 0 & , \; x < a \\ \dfrac{x-a}{b-a} & , \; a \le x \le b \\ 1 & , \; x > b \end{cases}$$

(a) Sketch **F(x)**

(b) Find the p·d·f of **X.** Hence show that the variance of **X** is $\frac{(b-a)^2}{12}$.

18. A **r·v X** is uniformly distributed with lower quartile **5** and upper quartile **9** in the interval **[a, b].** Find the**:**

(i) values of **a** and **b.**

(ii) $P(6 \le X \le 7)$

(iii) probability that **X** lies within one standard deviation of the mean**.**

(iv) cumulative distribution function of **X.**

[Ans: (i) 3, 11 (ii) 0·125 (iii) 0·5774]

19. The p·d·f of a r·v **X** is given by**:**

$$\mathbf{f(x)} = \begin{cases} \mathbf{\frac{1}{5}} & , \quad \mathbf{0 \le x \le 5} \\ \mathbf{0} & , \quad \mathbf{otherwise} \end{cases}$$

(i) Identify the distribution

(ii) Find $\mathbf{P(|X - \mu| < \sigma)}$ where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ is the mean and standard deviation of **X** respectively.

[Ans: (ii) 0·5774]

20. The life time in years of a battery is known to be uniformly distributed with mean **4** and variance $\mathbf{4/3}$ **,** issued with a three years guarantee. If two such batteries are picked at random, find the probability that both will be replaced under the guarantee.

[Ans: (ii) 0·0625]

21. A r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} \mathbf{3x^a} & , \quad \mathbf{0 \le x \le 1} \\ \mathbf{0} & , \quad \mathbf{otherwise} \end{cases}$$

Find the**:**

(i) value of **a.**

(ii) median of **X.**

[Ans: (i) 2 (ii) 0·7937]

22. A r·v **X** has the following cumulative distribution function

$$\mathbf{F(x)} = \begin{cases} \mathbf{0} & , \ \mathbf{x < 0} \\ \mathbf{\frac{1}{32}(x^4 - 8x^3 + \beta x^2)}, & \mathbf{0 \le x \le 4} \\ \mathbf{1} & , \ \mathbf{x > 4} \end{cases}$$

Find**:**

(i) the value of **β.**

(ii) P(X < 2)

(iii) the p·d·f of **X**

(iv) the mode of the distribution

[Ans: (i) 18 (ii) 0·75 (iv) 1]

23. The cumulative distribution of a continuous **r·v X** is such that**:**

$$\mathbf{F(x)} = \begin{cases} \mathbf{0} & , \ \mathbf{x \le 0} \\ \mathbf{0 \cdot 25x^2} & , \ \mathbf{0 < x \le 1} \\ \beta \mathbf{x} + \alpha & , \ \mathbf{1 < x \le 2} \\ \mathbf{0 \cdot 25(5-x)(x-1)}, & \mathbf{2 < x \le 3} \\ \mathbf{1} & , \ \mathbf{x > 3} \end{cases}$$

Find**:**

(i) the values of β and α**.**

(ii) $\mathbf{P(3 \le 2X \le 5)}$

(iii) the p·d·f of **X** and sketch it**.**

(iv) the mean, **μ** of **X.**

[Ans: (i) 0·5, –0 ·25 (ii) 0·4375 (iv) 1·5]

24. A continuous r·v **X** is distributed as follows**:**

$$\mathbf{P(X > x) = \alpha + \beta x^3 \quad , \quad 0 < x \le 3}$$

(i) Find the values of β and α.

(ii) Show that **E(X) = 2·25,** and find the standard deviation **σ** of **X.**

[Ans: (i) 1 , $\frac{-1}{27}$ (ii) 0·581]

25. The p·d·f of r·v **X** is given by

$$f(x) = \begin{cases} \beta x(16 - x^2) & , \quad 0 \le x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the**:**

(i) value of **β**

(ii) mode of **X**

(iii) mean of **X**

[Ans: (i) $\frac{1}{64}$ (ii) 2·3094 (iii) $\frac{32}{15}$]

26. A **r·v X** is uniformly distributed over the interval **[2, 6].** Find

(i) $\mathbf{P(X \ge 2\cdot9 / X \le 3\cdot5)}$

(ii) the variance of **X**

[Ans: (i) 0·4 (ii) $\frac{4}{3}$]

27. A continuous r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} 2 - 4x & , \quad 0 \le x \le 0\cdot25 \\ 1 & , \quad 0\cdot25 < x \le 0\cdot5 \\ 4x - 1 & , \quad 0\cdot5 < x \le 0\cdot75 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Sketch ***f(x),*** hence deduce the mean, **μ** and median of **X.**

(b) Find**:**

(i) the cumulative distribution function of **X.**

(ii) $P(|X - \mu| \le 0\cdot2)$

[Ans: (a) 0·375, 0·375 b(ii) 0·4225]

28. The cumulative distribution of a continuous **r·v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x \le 1 \\ \frac{1}{12}(x-1)^2 & , \quad 1 < x \le 3 \\ \frac{1}{24}(14x - x^2 - 25) & , \quad 3 < x \le 7 \\ 1 & , \quad x > 7 \end{cases}$$

Find**:**

(i) $P(2\cdot8 < X < 5\cdot2)$

(ii) the median of **X**

(iii) the p·d·f of **X** and sketch it**.**

(iv) the standard deviation of **X.**

[Ans: (i) 0·595 (ii) 3·45 (iv) 1·2472]

29. The number of boats **X** crossing a river is uniformly distributed between **30** and **110** boats**.** Find the**:**

(i) probability that at least **90** boats cross the river**.**

(ii) expected number of boats to cross the river**.**

(iii) standard deviation for the number of boats to cross the river**.**

(iv) probability that **X** lies within one standard deviation of the mean**.**

(v) upper quartile for the number of boats to cross the river**.**

(vi) 25^{th} percentile for the number of boats to cross the river**.**

(vii) cumulative distribution function of **X** and sketch it**.**

[Ans: (i) 0·25 (ii) 70 (iii) 23·094 (iv) 0·5774 (v) 90 (vi) 50]

30. The p·d·f of r·v **X** is given by

$$f(x) = \begin{cases} \beta x(3x - x^2) & , \quad 0 \le x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the**:**

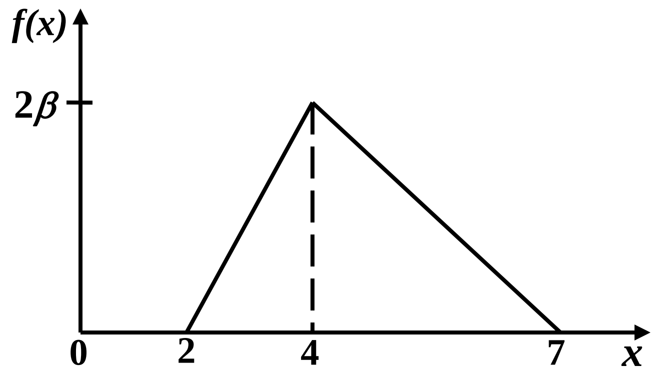
(i) value of **β**

(ii) mode of **X**

(iii) mean of **X**

[Ans: (i) $4/27$ (ii) 2 (iii) 1·8]

31. The p·d·f ***f(x)*** of a **r·v X** takes on the form shown in the sketch below**:**



Find the**:**

(i) value of $\boldsymbol{\beta}$

(ii) equations of the p·d·f

(iii) median of **X.**

[Ans: (i) 0·2 (iii) 4·2614]

32. The weekly demand for petrol **X** in thousands of units at the petrol station is given by the p·d·f

$$f(x) = \begin{cases} \beta x(\lambda - x) & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) Given that the mean weekly demand is **625** units, find the values of $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$. Hence obtain the mode of **X.**

(ii) If every week the petrol station stocks **750** units of petrol, find the probability that in a given week the petrol station will be unable to meet the demand for petrol.

(iii) Find the amount of petrol that should be stocked in order to be **85·05%** certain that the demand for petrol in that week will be met**.**

[Ans: (i) 1·5, 2 (ii) 0·3672 (iii) 900units]

33. A r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} \frac{2}{3a}(x+a) & , \ -a \le x \le 0 \\ \frac{1}{3a}(2a-x) & , \ 0 < x < 2a \\ 0 & , \ \text{otherwise} \end{cases}$$

Find**:**

(i) the value of **a**.

(ii) the expressions for $P(X \le x)$ and sketch it

(iii) the median of **X.**

(iv) $P(X \le 1{\cdot}5 / X \ge 0)$

[Ans: (i) 1 (iii) 0·2679 (iv) 0·9375]

34. A r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} \frac{1}{3a}(x+2) & , \ -a \le x \le 1 \\ \frac{1}{a}(2-x) & , \ 1 \le x \le a \\ 0 & , \ \text{otherwise} \end{cases}$$

Find**:**

(i) the value of **a**.

(ii) $P(X \le 0)$

(iii) the lower quartile of **X.**

[Ans: (i) 2 (iii) $^1/_3$ (iii) –0·2679]

35. A continuous r·v **X** has the following p·d·f

$$f(x) = \begin{cases} \lambda \sin x \ , & 0 \le x \le \pi \\ 0 \ , & \text{otherwise} \end{cases}$$

Find**:**

(i) the value of λ.

(ii) $P\left(X \ge \frac{\pi}{3}\right)$

(iii) the median of **X.**

[Ans: (i) 0·5 (ii) 0·75 (iii) $\frac{\pi}{2}$ **]**

36. A continuous r·v **X** has the following p·d·f

$$f(x) = \begin{cases} \lambda(1-\cos x) \ , & 0 \le x \le \pi/2 \\ \lambda \sin x \ , & \pi/2 \le x \le \pi \\ 0 \ , & \text{otherwise} \end{cases}$$

(a) Find**:**

(i) the value of λ.

(ii) $P\left(\frac{\pi}{3} < X < \frac{3\pi}{4}\right)$

(b) Show that the mean**,** μ of the distribution is $1 + \frac{\pi}{4}$.

[Ans: (i) $\frac{2}{\pi}$ **(ii) 0·6982]**

37. A r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} \lambda\cos x & , \quad 0 \le x \le \frac{\pi}{4} \\ \lambda\sin x & , \quad \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find**:**

(i) the value of λ**,** hence sketch ***f(x)*.**

(ii) $P\left(x \ge \frac{\pi}{4} \middle/ 0 \le x \le \frac{\pi}{3}\right)$

(iii) the mean of **X.**

[Ans: (i) $\frac{\sqrt{2}}{2}$ **(ii) 0·2265 (iii)** $\frac{\pi}{4}$ **]**

38. The cumulative distribution of a continuous **r·v X** is such that**:**

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \beta\sin^{-1}x & , \quad 0 \le x \le 1 \\ \lambda\tan^{-1}x & , \quad 1 \le x \le \sqrt{3} \\ 1 & , \quad x > \sqrt{3} \end{cases}$$

(a) Find**:**

(i) the values of λ and β**.**

(ii) $P(0{\cdot}5 < X < 1{\cdot}5)$

(iii) the p·d·f of **X.**

(b) Show that the mean, μ of the distribution is $\frac{3}{2\pi}[1 + \text{In}2]$

[Ans: (i) $\frac{3}{\pi}$, $\frac{3}{2\pi}$ **(ii) 0·6885]**

39. A r·v **X** has the following cumulative distribution function.

$$\mathbf{F(x)} = \begin{cases} \mathbf{0} & , \ \mathbf{x < 1} \\ \boldsymbol{\beta}\mathbf{(x^2 - x)} & , \ \mathbf{1 \le x \le 2} \\ \boldsymbol{\lambda}\mathbf{(x^2 + x)} & , \ \mathbf{2 < x \le 3} \\ \mathbf{1} & , \ \mathbf{x > 3} \end{cases}$$

Find**:**

(i) the values of λ and β.

(ii) the p·d·f of **X**

(iii) the mean**, μ** of **X**.

(iv) $\mathbf{P(|X - \mu| \le 0{\cdot}5)}$

[Ans: (i) $\mathbf{1/12}$**,** $\mathbf{1/4}$ **(ii) 2·0556 (iii) 0·5412]**

40. The p·d·f of a **r·v X** is given by**:**

$$\mathbf{f(x)} = \begin{cases} \dfrac{\boldsymbol{\lambda}}{\mathbf{x^2 + 1}} & , \ \mathbf{0 \le x \le 1} \\ \mathbf{0} & , \ \textbf{otherwise} \end{cases}$$

Show that the**:**

(i) value of $\boldsymbol{\lambda} = \dfrac{\mathbf{4}}{\boldsymbol{\pi}}$.

(ii) $\mathbf{P\left(X \ge \dfrac{1}{\sqrt{3}} \right) = \dfrac{1}{3}}$.

(iii) $\mathbf{E(X) = \dfrac{2In2}{\pi}}$.

(iv) median of the distribution is $\tan\dfrac{\boldsymbol{\pi}}{\mathbf{8}}$

41. The cumulative distribution of a continuous **r·v X** is given by**:**

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \beta \tan^{-1} x & , \quad 0 \le x \le \sqrt{3} \\ 1 & , \quad x > \sqrt{3} \end{cases}$$

(a) Find**:**

(i) the value of β**.**

(ii) P(X > 1)

(b) Show that the**:**

(i) median of **X** is $\tan\frac{\pi}{6}$**.**

(ii) 75th percentile of **X** is $\tan\frac{\pi}{4}$**.**

(c) By stating the p·d·f of **X,** show that $E(X) = \frac{3\text{In}2}{\pi}$**.**

42. The times of arrival of a bus at its stage are uniformly distributed between the interval **9:00am** to **2:00pm.** Find the**:**

(i) mean and variance of the bus's time of arrival

(ii) probability that the time of arrival does not exceed **1:00pm.**

[Ans: (i) 11·5h, $^{25}\!/_{12}$ **(iii) 0·8]**

43. A **r·v X** is uniformly distributed over the interval **[a, b].**

(a) State the p·d·f of **X** and sketch it**.**

(b) Show that the lower quartile of **X** is $\frac{3a + b}{4}$ and the upper is $\frac{a + 3b}{4}$**.**

44. A r·v **X** has the following p·d·f

$$f(x) = \begin{cases} \beta x & , \ 1 \le x \le 3 \\ \lambda(4-x) & , \ 3 < x \le 4 \\ 0 & , \ \text{otherwise} \end{cases}$$

(a) Sketch ***f(x)***

(b) Find**:**

(i) the values of β and λ

(ii) the mean of **X**

(iii) $P(3 \le X \le 4 / X > 2)$

[Ans: b(i) $2/11$, $6/11$ **(ii) 2·4848 (iii) 0·375]**

45. A continuous r·v **X** has the following p·d·f.

$$f(x) = \begin{cases} \frac{x}{3} - \frac{2}{3} & , \ 2 \le x \le 3 \\ \lambda & , \ 3 < x \le 5 \\ 2 + \beta x & , \ 5 < x \le 6 \\ 0 & , \ \text{otherwise} \end{cases}$$

(a) Find the values of λ and β

(b) Sketch ***f(x),*** hence deduce the mean, **μ** of **X.**

(c) Find the**:**

(i) variance of **X.**

(ii) E(3X) and **Var(3X)**

[Ans: (a) $1/3$, $-1/3$ **(b) 4 c(i)** $101/6$ **(ii) 12, 151·5]**

46. The mass **X kg** of maize flour produced per hour is modeled by a continuous r·v whose p·d·f is given by**:**

$$f(x) = \begin{cases} \lambda(4 - x^2) & , \quad 0 \le x \le 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Find the**:**

(i) value of λ

(ii) mean mass produced per hour

(b) Given that maize flour is sold at **£ 8** per kg and the cost of running the production is **£ 1** per hour, find the**:**

(i) expected profit per hour**.**

(ii) probability that in an hour the profit will exceed **£ 11.**

[Ans: a(i) $3/16$ (ii) $3/4$ b(i) £ 5 (ii) 0·0859]

47. A r·v **X** is uniformly distributed over the interval $a \le X \le b$**.** Given that **X** is distributed with mean **9** and variance **12,** find:

(i) the values of **a** and **b.**

(ii) $P(X \le 10)$

[Ans: (i) 3, 15 (ii) $7/12$]

48. A **r·v X** is uniformly distributed over the interval **[a, b].**

(a) State the p·d·f of **X** and sketch it**.**

(b) Show that $P(x_1 \le X \le x_2) = \dfrac{x_2 - x_1}{b - a}$

49. A continuous r·v **X** has the following p·d·f

$$f(x) = \begin{cases} \lambda x(3-x) & , \quad 0 \le x \le 2 \\ \lambda(4-x) & , \quad 2 \le x < 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find**:**

(i) the value of λ.

(ii) the mean of **X.**

(iii) the cumulative distribution function of **X**.

(iv) $P(1 \le X \le 3)$

[Ans: (i) $\frac{3}{16}$ (ii) 1·75 (iv) 0·6875]

50. The lifetime X in years of an electric bulb is a r·v **X** with the following p·d·f

$$f(x) = \begin{cases} \lambda x(5-x) & , \quad 0 \le x \le 5 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(a) Find the**:**

(i) value of λ.

(ii) mean of **X.**

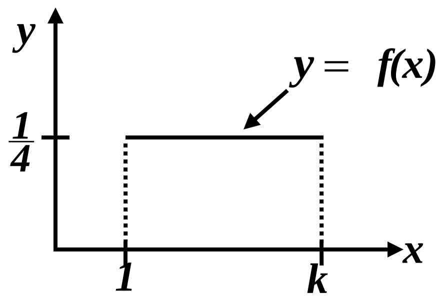
(b) If two such new bulbs are sold, find the probability that**:**

(i) both bulbs fail to work within one year.

(ii) only one bulb works for more than three years.

[Ans: (a) (i) $\frac{6}{125}$ (ii) 2·5 (b) (i) 0·0108 (ii) 0·4562]

51. A uniformly distributed **r·v *X*** has the following p·d·f ***f (x):***



Find the**:**

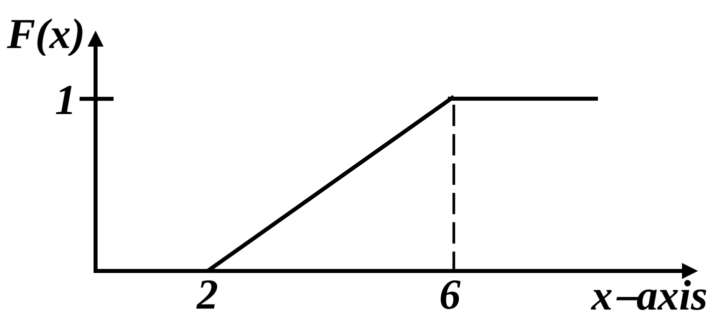
(i) value of ***k***

(ii) equations of the p·d·f of ***X***

(iii) variance of ***X***

[Ans: (i) 5 (iii) $\frac{4}{3}$]

52. The cumulative distribution function of a continuous ***r·v X*** is illustrated as follows:



Find**:**

(i) the p·d·f of **X** and sketch it**.**

(ii) the mean and variance of **X.**

(iii) $P(X \geq 3 / X < 5)$

[Ans: (ii) 4, $\frac{4}{3}$ (iii) $\frac{2}{3}$]